

UNIFORMLY EQUICONTINUOUS SETS, RIGHT MULTIPLIER TOPOLOGY, AND CONTINUITY OF CONVOLUTION

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1. INTRODUCTION

The dual space of the C^* -algebra $U_b(X)$ of bounded uniformly continuous complex-valued functions on a uniform space X carries several natural topologies. One of these is the topology of uniform convergence on bounded uniformly equicontinuous sets, or the UEB topology for short. In the particular case when $X = G$ is a topological group and $U_b(X) = LUC(G)$, the C^* -algebra of bounded left uniformly continuous functions, the UEB topology plays a significant role in the continuity of the convolution product on $LUC(G)^*$. In this paper we derive a useful characterisation of bounded uniformly equicontinuous sets on locally compact groups. Then we demonstrate that for every locally compact group G the UEB topology on the space $M(G)$ of finite Radon measures on G coincides with the right multiplier topology, $M(G)$ viewed as the multiplier algebra of the L^1 group algebra. In this sense the UEB topology is a generalisation to arbitrary topological groups of the multiplier topology for locally compact groups. In the final section we prove results about UEB continuity of convolution on $LUC(G)^*$ (even for topological groups not necessarily locally compact).

Many of our proofs use factorisation techniques such as Cohen's factorisation theorem (see for example [1, Theorem I.11.10]). In fact some of our results may be interpreted as strengthenings of Cohen's factorisation. Theorem 1 gives a factorisation of a bounded equicontinuous set of functions in $LUC(G)$ by a single function in $L^1(G)$. We shall also need the factorisation theorem due to Neufang [7].

2. PRELIMINARIES

Let X be a uniform space and let $U_b(X)$ be the C^* -algebra of bounded uniformly continuous complex-valued functions on X . Say that $\mathcal{F} \subseteq U_b(X)$ is a *UEB set* if it is bounded in the sup norm and uniformly equicontinuous. Say that a net (m_α) in $U_b(X)^*$ converges to 0 in the *UEB topology* if

$$\sup_{f \in \mathcal{F}} |\langle m_\alpha, f \rangle| \rightarrow 0$$

for every UEB set $\mathcal{F} \subseteq U_b(X)$.

For a topological group G (not necessarily locally compact), $LUC(G)$ is the space of the bounded functions f on G such that for every $\epsilon > 0$ there is a neighbourhood U of the identity in G with

$$|f(s) - f(t)| < \epsilon \quad \text{whenever } st^{-1} \in U.$$

Say that a set $\mathcal{F} \subseteq \text{LUC}(G)$ is *equi-LUC* if \mathcal{F} is bounded in the sup norm and for every $\epsilon > 0$ there is a neighbourhood U of the identity in G such that

$$|f(s) - f(t)| < \epsilon \quad \text{whenever } f \in \mathcal{F}, st^{-1} \in U.$$

It is well known that $\text{LUC}(G)$ is the space of the bounded functions that are uniformly continuous with respect to a suitable uniformity on G (the *right uniformity* in the terminology of Hewitt and Ross [5]). In the following we consider G as a uniform space with this uniformity, so that $\text{LUC}(G) = \text{U}_b(G)$ and UEB sets are exactly the equi-LUC sets.

Equi-LUC subsets of $\text{LUC}(G)$ arise naturally in the study of convolution on the dual $\text{LUC}(G)^*$ of $\text{LUC}(G)$. On general uniform spaces, UEB sets of functions have an important role in the theory of so called uniform measures — see [8][9] and the references there.

We identify the space $\text{M}(G)$ of finite Radon measures on a topological group G with a subspace of $\text{LUC}(G)^*$ by means of the duality $\langle m, f \rangle = \int f dm$ for $m \in \text{M}(G)$ and $f \in \text{LUC}(G)$. (We shall use the brackets $\langle \cdot, \cdot \rangle$ to denote Banach space duality in other cases too.)

Denote by $L_x f \in \text{LUC}(G)$ and $R_x f \in \text{LUC}(G)$ the left and the right translate of $f \in \text{LUC}(G)$ by $x \in G$; that is, $L_x f(s) = f(xs)$ and $R_x f(s) = f(sx)$ for $s \in G$. More generally, for $n \in \text{LUC}(G)^*$, denote by $R_n f$ the function $s \mapsto n(L_s f)$. Obviously $R_n = R_x$ when n is the point mass at $x \in G$.

Note that $R_n f \in \text{LUC}(G)$ whenever $n \in \text{LUC}(G)^*$ and $f \in \text{LUC}(G)$ (this well-known fact is also a special case of Lemma 6 below). It follows that we may define the *convolution* $m \star n \in \text{LUC}(G)^*$ for $m, n \in \text{LUC}(G)^*$ by

$$m \star n(f) = m(R_n f) \quad \text{for } f \in \text{LUC}(G).$$

Csiszár [4] and Lau [6], among others, have studied this notion of convolution on $\text{LUC}(G)^*$. It is also a particular case of the general notion of *evolution* as defined by Pym [10]. When $m, n \in \text{M}(G)$, this definition of $m \star n$ agrees with the standard definition of convolution of measures [5, 19.1]. We note that the spectrum G^{LUC} of $\text{LUC}(G)$ is closed under the convolution product of $\text{LUC}(G)^*$.

For the remainder of this section, suppose that G is a locally compact group, and denote by dt its left Haar measure. We identify $L^1(G)$ with the measures in $\text{M}(G)$ that are absolutely continuous with respect to dt , and the convolution in $\text{LUC}(G)^*$ restricted to $L^1(G)$ agrees with the usual convolution of functions as defined in [5, 20.10]:

$$f \star g(s) = \int f(st)g(t^{-1}) dt \quad (f, g \in L^1(G), s \in G).$$

The same formula defines also the convolution $f \star g$ for $f \in L^1(G)$ and $g \in L^\infty(G)$ ([5, 20.16]).

The dual space of $L^1(G)$ is identified with the space $L^\infty(G)$ of essentially bounded measurable functions on G . The convolution on $L^1(G)$ defines the right action of $L^1(G)$ on $L^\infty(G)$ via

$$\langle h \cdot f, g \rangle = \langle h, f \star g \rangle \quad (f, g \in L^1(G), h \in L^\infty(G)).$$

Defining $\tilde{f}(s) = f(s^{-1})/\Delta(s)$ for $f \in L^1(G)$, where Δ is the modular function of G , we can write $h \cdot f = \tilde{f} \star h$. It then follows from Cohen's factorisation theorem that $\text{LUC}(G) = L^\infty(G) \cdot L^1(G) = L^1(G) \star L^\infty(G)$.

Next we have the left action of $L^\infty(G)^*$ on $L^\infty(G)$:

$$\langle m \cdot h, f \rangle = \langle m, h \cdot f \rangle \quad (f \in L^1(G), h \in L^\infty(G), m \in L^\infty(G)^*).$$

In fact this action depends only on the restriction of m to $\text{LUC}(G) \subseteq L^\infty(G)$ (since $h \cdot f \in \text{LUC}(G)$), so this also gives an action of $\text{LUC}(G)^*$ on $L^\infty(G)$, which we denote the same way. Moreover, the action leaves $\text{LUC}(G)$ invariant. It is easily seen that in fact $m \cdot h = R_m h$ for every $h \in \text{LUC}(G)$ and $m \in \text{LUC}(G)^*$.

Finally, we define the left Arens product on $L^\infty(G)^*$ by

$$\langle n \cdot m, h \rangle = \langle n, m \cdot h \rangle \quad (h \in L^\infty(G), m, n \in L^\infty(G)^*).$$

It follows from the remarks in the preceding paragraph that $\text{LUC}(G)^*$ equipped with convolution is a quotient Banach algebra of $L^\infty(G)^*$ equipped with the left Arens product.

3. CHARACTERISATION OF EQUI-LUC SETS ON LOCALLY COMPACT GROUPS

The following result characterises equi-LUC sets on locally compact groups as the bounded sets that can be simultaneously factorised by one function in $L^1(G)$.

Theorem 1. *Let G be a locally compact group and let \mathcal{F} be a set of functions on G . Then \mathcal{F} is equi-LUC if and only if there is $f \in L^1(G)$ and a bounded set $\mathcal{H} \subseteq \text{LUC}(G)$ such that*

$$\mathcal{F} = f \star \mathcal{H}.$$

It is easy to check that every set of the form $f \star \mathcal{H}$ with $f \in L^1(G)$ and bounded $\mathcal{H} \subseteq \text{LUC}(G)$ is equi-LUC, so one direction of the theorem is clear.

We shall prove the converse separately for compact groups and for non-compact ones. First we shall derive the compact case from a general factorisation result for norm-compact subsets of left Banach A -modules. Then we prove the non-compact case using a different factorisation result due to Neufang [7]. It is noteworthy that the two factorisation results are based on opposite ideas: the non-compact case relies heavily on non-compactness and the compact case on compactness.

We base the compact case to the following theorem. It is actually a known result due to Craw [3, Corollary], but we present a different proof using Cohen's factorisation theorem directly. Craw obtained his result as a corollary to a factorisation theorem for null sequences in Fréchet algebras (similar to Varopoulos's factorisation of null sequences in Banach algebras). In our case, the Varopoulos result is an immediate corollary.

Theorem 2. *Let A be a Banach algebra and X a left Banach A -module. Suppose that A has a bounded approximate identity for the action on X . If K is a norm-compact subset of X then there is a in A such that for every x in K there is y in X such that*

$$x = ay.$$

Proof. A routine argument shows that the collection \mathcal{S} of totally bounded double sequences on X forms a Banach space. We define an action of A on \mathcal{S} coordinate-wise:

$$a(y_{n,m}) = (ay_{n,m}).$$

The action makes \mathcal{S} a left Banach A -module. A bounded approximate identity (e_i) for the action of A on X is also a bounded approximate identity for the action on \mathcal{S} . Indeed, given $(y_{n,m}) \in \mathcal{S}$ and $\epsilon > 0$, choose x_1, \dots, x_k in X such that ϵ -balls

centred at x_j 's cover $(y_{n,m})$. Then we can choose i_0 such that $\|e_i x_j - x_j\| < \epsilon$ for every $i \geq i_0$ and $j = 1, \dots, k$. Choosing a suitable x_j for each $y_{n,m}$ we have

$$\|e_i y_{n,m} - y_{n,m}\| \leq \|e_i y_{n,m} - e_i x_j\| + \|e_i x_j - x_j\| + \|x_j - y_{n,m}\| \leq \|e_i\| \epsilon + 2\epsilon$$

for every $i \geq i_0$. So (e_i) is a bounded approximate identity also for the action on \mathcal{S} .

Since K is norm-compact, there is a double sequence $(x_{n,m})$ in K such that for every x in K ,

$$\|x - x_{n,m}\| < 1/n$$

for some m . Then $(x_{n,m}) \in \mathcal{S}$, so we can apply Cohen's factorisation to find $a \in A$ and $(y_{n,m}) \in \mathcal{S}$ such that

$$x_{n,m} = a y_{n,m}.$$

Given $x \in K$ there is, by construction, a sequence $(x_{n,m_n})_{n=1}^\infty$ that converges to x . By factorisation $x_{n,m_n} = a y_{n,m_n}$. The sequence (y_{n,m_n}) is totally bounded and hence has a subsequence converging to some y in X . It follows that $x = ay$. \square

Proof of the compact case of Theorem 1. Due to compactness of G an equi-LUC set on G is totally bounded (i.e. precompact), so its closure is a norm-compact subset of $\text{LUC}(G)$. Since $\text{LUC}(G)$ is a left Banach $L^1(G)$ -module and $L^1(G)$ has a bounded approximate identity for the action, we may apply Theorem 2. \square

To prove the non-compact case of Theorem 1, it is convenient to use the notation of Arens actions introduced in Section 2.

Proof of the non-compact case of Theorem 1. Let $\mathcal{F} = \{f_i\}$ be an equi-LUC set of functions on a non-compact group G . By Lemma 2.2 of [7], we have a factorisation

$$f_i = x_i \cdot h$$

where $h \in \text{LUC}(G)$ and $\{x_i\} \subseteq G^{\text{LUC}}$. By Cohen's factorisation theorem, we can factorise $h = g \cdot u$ where $u \in L^1(G)$ and $g \in \text{LUC}(G)$. Then

$$x_i \cdot h = (x_i \cdot g) \cdot u = \tilde{u} \star (x_i \cdot g).$$

Putting $h_i = x_i \cdot g$ and $f = \tilde{u}$ we get that for every i

$$f_i = f \star h_i.$$

This is the required factorisation. \square

4. UEB TOPOLOGY AND THE RIGHT MULTIPLIER TOPOLOGY

In this section we specialise to the case of locally compact groups and show that on the measure algebra $M(G) \subseteq \text{LUC}(G)^*$ the UEB topology coincides with the right multiplier topology. Here we view $M(G)$ as the algebra of right multipliers of $L^1(G)$ (through Wendel's theorem [13]). The *right multiplier topology* is the topology induced by the seminorms $m \mapsto \|f \star m\|$ where f runs through the elements of $L^1(G)$.

Theorem 3. *Let G be a locally compact group. The UEB topology on $\text{LUC}(G)^*$ coincides with the topology generated by the seminorms $m \mapsto \|f \star m\|$, $f \in L^1(G)$. In particular, the right multiplier topology of $M(G)$ agrees with the UEB topology inherited from $\text{LUC}(G)^*$.*

Proof. This is rather immediate from the factorisation result in Theorem 1. Suppose first that $m_\alpha \rightarrow 0$ in $\text{LUC}(G)^*$ with respect to the UEB topology. Let B denote the unit ball of $\text{LUC}(G)$. Then, for every f in $L^1(G)$,

$$\|f \star m_\alpha\| = \sup_{h \in B} |\langle f \star m_\alpha, h \rangle| = \sup_{h \in B} |\langle m_\alpha, \tilde{f} \star h \rangle|.$$

Since $\tilde{f} \star B$ is an equi-LUC set, it follows that $\|f \star m_\alpha\| \rightarrow 0$.

Conversely, suppose that $m_\alpha \rightarrow 0$ in the topology generated by the seminorms $m \mapsto \|f \star m\|$, $f \in L^1(G)$. Now if \mathcal{F} is an equi-LUC set, then by Theorem 1 we have $\mathcal{F} = f \star \mathcal{H}$ with $f \in L^1(G)$ and $\mathcal{H} \subseteq \text{LUC}(G)$ bounded. Therefore

$$\sup_{g \in \mathcal{F}} |\langle m_\alpha, g \rangle| = \sup_{h \in \mathcal{H}} |\langle \tilde{f} \star m_\alpha, h \rangle| \leq \|\tilde{f} \star m_\alpha\| \sup_{h \in \mathcal{H}} \|h\| \rightarrow 0.$$

□

5. CONTINUITY OF CONVOLUTION IN $\text{LUC}(G)^*$

In this section we prove several results that illustrate the role of equi-LUC sets in the study of convolution on the dual $\text{LUC}(G)^*$ of $\text{LUC}(G)$. There are more results of this kind, presented in a more general framework, in [8] and [9].

Lemma 4. *Let G be a topological group, $m \in M(G)$, and let \mathcal{F} be an equi-LUC subset of $\text{LUC}(G)$. Then the restriction of m to \mathcal{F} is continuous in the G -pointwise topology.*

Proof. The G -pointwise topology and the compact-open topology coincide on every equi-LUC subset of $\text{LUC}(G)$. □

Lemma 5. *Let G be a topological group and $\mathcal{F} \subseteq \text{LUC}(G)$ an equi-LUC set.*

- (1) *The set $\{R_x f \mid f \in \mathcal{F}, x \in G\}$ is equi-LUC.*
- (2) *For every $x \in G$ the set $\{L_x f \mid f \in \mathcal{F}\}$ is equi-LUC.*
- (3) *If G is a SIN group then the set $\{L_x f \mid f \in \mathcal{F}, x \in G\}$ is equi-LUC.*

Proof. Part (1) follows directly from the definition of equi-LUC sets. Part (2) follows from the continuity of group operations: If U is a neighbourhood of the identity in G , then so is xUx^{-1} for every $x \in G$. Part (3) follows from part (1) by the left-right symmetry. □

The next lemma generalises part (1) of Lemma 5.

Lemma 6. *Let G be a topological group, $\mathcal{F} \subseteq \text{LUC}(G)$ an equi-LUC set, and $B \subseteq \text{LUC}(G)^*$ a norm-bounded set. Then the set $\{R_n f \mid f \in \mathcal{F}, n \in B\}$ is equi-LUC.*

Proof. Write $c = \sup\{\|n\| \mid n \in B\}$. For $f \in \mathcal{F}$, $n \in B$ and $s \in G$ we have

$$|R_n f(s)| = |n(L_s f)| \leq \|n\| \cdot \|L_s f\| = \|n\| \cdot \|f\| \leq c\|f\|,$$

so that the set $\{R_n f \mid f \in \mathcal{F}, n \in B\}$ is bounded in the sup norm.

Take any $\epsilon > 0$, and let U be a neighbourhood of the identity in G such that $|f(s) - f(t)| < \epsilon$ whenever $f \in \mathcal{F}$, $st^{-1} \in U$. If $st^{-1} \in U$ and $f \in \mathcal{F}$, then

$$|L_s f(x) - L_t f(x)| = |f(sx) - f(tx)| < \epsilon \text{ for } x \in G$$

and

$$|R_n f(s) - R_n f(t)| = |n(L_s f - L_t f)| \leq \|n\| \cdot \|L_s f - L_t f\| \leq c\epsilon$$

which completes the proof. □

With the convolution product $\text{LUC}(G)^*$ is a Banach algebra, and in particular the convolution is jointly norm-continuous on $\text{LUC}(G)^*$. On the other hand, the convolution is far from being jointly continuous in the $\text{LUC}(G)$ -weak topology. By Theorem 1 in [12], if G is not compact then the convolution is not jointly $\text{LUC}(G)$ -weakly continuous even when restricted to bounded sets in $\text{M}(G)$.

Nevertheless, we now show that the convolution on $\text{M}(G)$ is well-behaved in the UEB topology. Recall from section 1 that this is the topology on $\text{LUC}(G)^*$ of uniform convergence on the equi-LUC subsets of $\text{LUC}(G)$.

The proof of the following theorem does not use any properties of Radon measures other than Lemma 4. Thus the same proof establishes a more general result in which $\text{M}(G)$ is replaced by the space of uniform measures. For this and other generalisations, see [8][9].

Theorem 7. *Let G be a topological group, let $B \subseteq \text{LUC}(G)^*$ be a norm-bounded set, $m_0 \in \text{LUC}(G)^*$ and $n_0 \in B$. Then the mapping $(m, n) \mapsto m \star n$ from $\text{LUC}(G)^* \times B$ to $\text{LUC}(G)^*$ is jointly continuous at (m_0, n_0) in the UEB topology in each of these two cases:*

- (a) $m_0 \in \text{M}(G)$,
- (b) G is a SIN group.

For commutative groups, a result similar to case (a) (for uniform measures) was proved by Caby [2]. Other theorems of this kind were also proved by Csiszár [4, Th.1] and Pym [11, 2.2]. (There is a gap in the proof of Theorem 1 in [4]; to correct it, a boundedness condition should be added to the statement of the theorem.)

Proof. Let $(m_\alpha)_{\alpha \in A}$ be a net in $\text{LUC}(G)^*$ and $(n_\alpha)_{\alpha \in A}$ a net in B . Let $m_\alpha \rightarrow m_0$ and $n_\alpha \rightarrow n_0$ in the UEB topology. Take any equi-LUC set $\mathcal{F} \subseteq \text{LUC}(G)$.

Define $g_\alpha(x) = \sup\{|R_{n_\alpha}f(x) - R_{n_0}f(x)| \mid f \in \mathcal{F}\}$ for $\alpha \in A$, $x \in G$. It follows from Lemma 6 that the set $\{g_\alpha \mid \alpha \in A\}$ is equi-LUC. On the other hand, for every $x \in G$ the set $\{L_x f \mid f \in \mathcal{F}\}$ is equi-LUC by Lemma 5, and since $n_\alpha \rightarrow n_0$ in the UEB topology, $g_\alpha(x) \rightarrow 0$.

Now

$$\begin{aligned} |m_\alpha \star n_\alpha(f) - m_0 \star n_0(f)| &= |m_\alpha(R_{n_\alpha}f) - m_0(R_{n_0}f)| \\ &\leq |m_\alpha(R_{n_\alpha}f) - m_0(R_{n_\alpha}f)| + |m_0(R_{n_\alpha}f) - m_0(R_{n_0}f)|. \end{aligned}$$

The set $\{R_{n_\alpha}f \mid f \in \mathcal{F}, \alpha \in A\}$ is equi-LUC by Lemma 6, so

$$\lim_{\alpha} \sup_{f \in \mathcal{F}} |m_\alpha(R_{n_\alpha}f) - m_0(R_{n_\alpha}f)| = 0.$$

Therefore,

$$\begin{aligned} \lim_{\alpha} \sup_{f \in \mathcal{F}} |m_\alpha \star n_\alpha(f) - m_0 \star n_0(f)| &\leq \lim_{\alpha} \sup_{f \in \mathcal{F}} |m_\alpha(R_{n_\alpha}f) - m_0(R_{n_\alpha}f)| \\ &\quad + \lim_{\alpha} \sup_{f \in \mathcal{F}} |m_0(R_{n_\alpha}f) - m_0(R_{n_0}f)| \\ &\leq \lim_{\alpha} |m_0|(g_\alpha) = 0 \end{aligned}$$

where the last equality holds by Lemma 4 in case (a). In case (b) we have $\|g_\alpha\| \rightarrow 0$ by part 3 of Lemma 5, and thus again $|m_0|(g_\alpha) \rightarrow 0$. \square

Corollary 8. *Convolution is jointly UEB continuous on bounded subsets of $\text{M}(G)$ for every topological group G .*

Corollary 9. *Convolution is jointly UEB continuous on bounded subsets of $\text{LUC}(G)^*$ for every SIN group G .*

It is known [8][9] that in $M(G)$ the UEB topology and the $\text{LUC}(G)$ -weak topology coincide on the positive cone, on $\|\cdot\|$ spheres and on $\text{LUC}(G)$ -weakly compact sets. Thus from Corollary 8 we immediately obtain that the convolution is jointly $\text{LUC}(G)$ -weakly continuous when restricted to the positive cone or a $\|\cdot\|$ sphere or a $\text{LUC}(G)$ -weakly compact set in $M(G)$. In particular, \star is jointly $\text{LUC}(G)$ -weakly sequentially continuous on $M(G)$.

Example. In this example we exhibit a topological group G such that the convolution is not jointly UEB continuous on bounded subsets of $\text{LUC}(G)^*$. In particular, the statement of Theorem 7 is not true if both conditions (a) and (b) are removed. In fact, for this group G there are $m_0 \in G^{\text{LUC}}$ and a sequence of elements $y_j \in G$ that UEB converges to $y \in G^{\text{LUC}}$, and yet $m_0 \star y_j$ does not even converge to $m_0 \star y$ in the $\text{LUC}(G)$ -weak topology.

Let G be the group of increasing homeomorphisms of the interval $[0, 3]$ onto itself. The group operation is composition of maps. The topology of G is induced by the right-invariant metric

$$\Delta(x, y) := \sup_{t \in [0, 3]} |x(t) - y(t)| \quad \text{for } x, y \in G.$$

Define $y_j \in G$ for $j = 1, 2, \dots$ by

$$y_j(t) := \begin{cases} t/j & \text{when } 0 \leq t \leq 2 \\ \frac{2}{j} + (3 - \frac{2}{j})(t - 2) & \text{when } 2 < t \leq 3. \end{cases}$$

The sequence $\{y_j\}_j$ is Cauchy in the metric, hence it has a UEB limit $y \in G^{\text{LUC}}$.

Define the function $h \in \text{LUC}(G)$ by $h(x) := x(1)$ for $x \in G$. Let m_0 be a cluster point of the sequence (y_k^{-1}) in G^{LUC} . Then

$$h(y_k^{-1} \circ y_j) = y_k^{-1}(y_j(1)) = y_k^{-1}(1/j) = 2 + \frac{\frac{1}{j} - \frac{2}{k}}{3 - \frac{2}{k}}$$

for all $k \geq 2j$, and so

$$m_0 \star y_j(h) = \lim_k h(y_k^{-1} \circ y_j) = 2 + 1/3j.$$

On the other hand,

$$R_y h(x) = y(L_x h) = \lim_j L_x h(y_j) = \lim_j h(x \circ y_j) = \lim_j x(y_j(1)) = 0$$

for all $x \in G$, and so

$$m_0 \star y(h) = m_0(R_y h) = 0.$$

Hence $m_0 \star y(h) \neq \lim_j m_0 \star y_j(h)$. □

We do not know whether Corollary 8 holds without the boundedness restriction; that is, whether convolution is jointly UEB continuous on $M(G)$ for every topological group G . However, we now show that convolution is jointly UEB continuous even on $\text{LUC}(G)^*$ for every locally compact SIN group G . The proof relies on the characterisation of equi-LUC sets in section 3.

Theorem 10. *For every locally compact SIN group G , convolution is jointly UEB continuous on $\text{LUC}(G)^*$.*

Proof. By Theorem 3 it is enough to show that the convolution on is jointly continuous with respect to the topology generated by the seminorms $m \mapsto \|f \star m\|$, $f \in L^1(G)$. Let $m_\alpha \rightarrow m$ and $n_\alpha \rightarrow n$ be nets converging in this topology.

Let $ZL^1(G)$ denote the centre of $L^1(G)$. Then $ZL^1(G)$ consists of all central functions in $L^1(G)$ (i.e. functions $f \in L^1(G)$ such that $f(st) = f(ts)$ for almost every $s, t \in G$). Since $L^1(G)$ is weak*-dense in $LUC(G)^*$ and the convolution from left by an element in $L^1(G)$ is weak*-continuous on $LUC(G)^*$, it follows that $ZL^1(G)$ is contained in the centre of $LUC(G)^*$ as well. Now $L^1(G)$ has a bounded approximate identity consisting of central functions because G is SIN. Hence by Cohen factorisation theorem, $L^1(G) = L^1(G) \star ZL^1(G)$.

Let $f \in L^1(G)$ be arbitrary and factorise $f = g \star h$ where $g \in L^1(G)$, $h \in ZL^1(G)$. Then

$$f \star (m_\alpha \star n_\alpha) = (g \star m_\alpha) \star (h \star n_\alpha)$$

because $h \in ZL^1(G)$. Since $g \star m_\alpha \rightarrow g \star m$ and $h \star n_\alpha \rightarrow h \star n$ in norm, it follows that $f \star (m_\alpha \star n_\alpha) \rightarrow f \star (m \star n)$ in norm, as required. \square

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